SYDE 312 UNIT 4:
EXTRA QUADRATURE PROBLEMS

Problem E1

$$I = \int_{-1}^{1} \frac{dx}{x^2\sqrt{x^2+1}}$$

Use $f(x) = 1/x^2\sqrt{x^2+1}$

You can't use the Matlab gaussQuad file in the provided form, because it implements composite Gauss-Legendre quadrature with multiple panels. I've provided a different function on the course website (GLQ.m) which implements single interval quadrature in a simple way, with the capability to specify the number of nodes. Using that function (or looking the nodes and weights up and multiplying by hand), we get these values for the integral (6 decimals):

2-point: 5.196152
4-point: 11.394401
6-point: 17.651588

You can only use an even number of nodes, because otherwise a zero node is included, at which the function has a singularity and cannot be evaluated. This is a strange problem, because the integral actually diverges (Matlab quad gives NaN). Symbolic evaluation shows the two integrals over $[-1, 0]$ and $[0, 1]$ diverge. The sequence of values for our problem, using increasing (even) numbers of nodes, will not converge, but the integral stays finite because you only add finite numbers when you avoid the singularity as a node. Adaptive methods (like quadl) will adjust the interval size when you get near the singularity - either it will converge with minimal steps, or diverge, but they still need a lot of nodes near singularities. Explore these ideas with $\int_0^1 dx/\sqrt{(x)} = 2$. To get six decimals requires 469 nodes in quadl. Compare to plain GLQ, which gives 1.998260 with 500 nodes.

Problem E2

$$\begin{aligned} I &= \int_0^\infty x^2 e^{-x^2} dx \\ &= \int_0^\infty e^{-x} x^2 e^x e^{-x^2} dx \\ &= \int_0^\infty e^{-x} f(x) dx \end{aligned}$$

So we can use Gauss-Laguerre quadrature with $f(x) = x^2 e^{x-x^2}$. The results obtained from the provided function gaussLagQuad are:

2-point: 0.373774
3-point: 0.231979
4-point: 0.374374
5-point: 0.488092
6-point: 0.500577
40-point: 0.443113 (good to 6 decimals)

The standard Matlab quad and quadl functions cannot evaluate integrals with infinite limits. You have several choices, all of which require more than just turning the handle:

- pre-process the integal by manipulating it symbolically to remove the infinite limit (integration by parts, substitution etc.)
- use one of a clever Gaussian quadrature rules based on an infinite integration interval (e.g. Gauss-Laguerre)
- use a series of integrals over finite intervals, evaluated say with quad function, covering an increasingly larger (but still finite) integration interval: $\int_a^\infty = \int_a^{b_1} + \int_{b_1}^{b_2} + \int_{b_2}^{b_3} + \cdots$, truncating it after you reach a desired tolerance.
- evaluate $\int_a^b$ for increasing values of b until the difference between successive integrals is less than the desired tolerance

There is no easy solution in general to infinite limits.

Problem E3

$$I = \int_1^3 \frac{dx}{x^2(100-x^2)^{3/2}}$$

Exact value of the integral:

$$\begin{aligned} I &= \frac{1}{10000}\left[-\frac{\sqrt{100-9}}{3} + \frac{3}{\sqrt{100-9}} + \frac{\sqrt{100-1}}{1} - \frac{1}{\sqrt{100-1}}\right] \\ &= 6.98405784 \times 10^{-4} \end{aligned}$$

Now evaluate with Gauss-Legendre quadrature. Using the GLQ (see note in Problem E1 above) function I get:

2-point: 6.76366575E-04
4-point: 6.98188459E-04
6-point: 6.984042166E-04
10-point: 6.9840587017E-04

Matlab quad (tol set to 1E-10) gives: 6.9840587025E-04.

Problem E4

$$I = \int_2^\infty \frac{dx}{(x-1)^2} = 1$$

The values of the integral over finite intervals: $I_b = \int_2^b \frac{dx}{(x-1)^2}$ with different values of the upper limit of integration are shown below. In each case, the interval $[2, b]$ is divided into 16 segments for the application of Simpson's 1/3 rule. It can be seen that the results overshoot the exact value of the integral when $b > 20$, due to round off and truncation errors.

b	I_b
3	0.50000209
4	0.66669732
5	0.75014716
6	0.80043286
7	0.83430648
8	0.85898989
9	0.87812066
10	0.89373487
11	0.90706187
12	0.91888577
13	0.92972714
14	0.93994373
15	0.94978613
16	0.95943308
17	0.96901351
18	0.97862142
19	0.98832494
20	0.99817365
21	1.0082031
22	1.018439

Problem E5

$$K_1(m) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - m \sin^2 x}}$$

for $m = 0.5$.

The Matlab code:

```
f=inline('1./sqrt(1-0.5.*(sin(x).^2))');
q=quad(f,0,pi/2)
```

gives $K_1(0.5) = 1.85407469486752$

Taking a lower tolerance value than defualt: `q=quad(f,0,pi/2,1E-10)` gives: $K_1(0.5) = 1.85407467729916$

Problem E6

$$K_2(m) = \int_0^{\pi/2} \sqrt{1 - m\sin^2 x}\; dx$$

for $m = 0.5$. The Matlab code:

```
f=inline('sqrt(1-0.5.*(sin(x).^2))');
q=quad(f,0,pi/2,1e-10)
```

gives $K_2(0.5) = 1.35064388104617$

Incidentally, plain Gauss-Legendre quadrature is pretty good with this integral. Even just 4-point gives (using the GLQ function): $K_2(0.5) = 1.3506436986$, i.e. good to six decimals.

Problem E7

(a) First, using symbolic integration:

$$F = \int_0^2 180\sqrt{4 - x^2}xdx$$

Let $y = x^2, dy = 2xdx, xdx = dy/2$. Then

$$\begin{aligned} F &= \int_0^4 180\sqrt{4-y}\frac{dy}{2} \\ &= 90\int_0^4 \sqrt{4-y}dy \\ &= 90\left[-\frac{2}{3}(4-y)^{3/2}\right]_0^4 \\ &= 480.0 \end{aligned}$$

Now using the various numerical procedures. You should be able to apply the methods and get the following answers:

(b) Trapezoid rule with 12 steps: F= 469.11276

(c) Simpson's one-third rule with 12 steps: F=475.95059

(d) Simpson's 3/8 rule with 12 steps: F= 475.05002
(e) Gauss-Legendre quadrature: $a = 0, b = 2, x = (2t + 2)/2 = t + 1$

$$F = \int_0^2 180\sqrt{4-(x)^2}xdx = \int_{-1}^1 180\sqrt{4-(t+1)^2}(t+1)dt$$

4-point: 482.49299
5-point: 481.3285
6-point: 480.7925
8-point: 480.3489
20-point: 480.0244265

(f) The Matlab quad and quadl functions. Here's the code:

```
g=inline('180.*sqrt(4.-x.^2).*x');
quad(g,0,2);
```

$F = 479.9999963$

```
g=inline('180.*sqrt(4.-x.^2).*x');
quad(g,0,2,1e-10);
```

$F = 479.9999999996$

```
g=inline('180.*sqrt(4.-x.^2).*x');
quadl(g,0,2,1e-10);
```

$F = 479.9999999413$